

AF TECHNICAL REPORT NO. 5790

July 1949

**TRANSONIC FLOW  
THROUGH NEARLY CIRCULAR DUCTS  
WITH GRADUALLY VARYING CROSS SECTION.**

**Don Blose**

**DISTRIBUTION STATEMENT A**  
Approved for Public Release  
Distribution Unlimited

**UNITED STATES AIR FORCE  
AIR MATERIEL COMMAND  
WRIGHT-PATTERSON AIR FORCE BASE, DAYTON, OHIO**

**Reproduced From  
Best Available Copy**

**20000515 045**

### NOTE

When drawings, specifications, and other data prepared by the War Department are furnished to manufacturers and others for use in the manufacture or purchase of supplies, or for any other purpose, the Government assumes no responsibility nor obligation whatever; and the furnishing of said data by the War Department is not to be regarded by implication or otherwise, or in any manner licensing the holder, or conveying any rights or permission to manufacture, use, or sell any patented inventions that may in any way be related thereto.

---

The information furnished herewith is made available for study upon the understanding that the Government's proprietary interests in and relating thereto shall not be impaired. It is desired that the Patent & Royalties Section, Office of the Judge Advocate, Air Materiel Command, Wright-Patterson AFB, Dayton, Ohio, be promptly notified of any apparent conflict between the Government's proprietary interests and those of others.

---

### Espionage Act

Notice: "This document contains information affecting the national defense of the United States within the meaning of the Espionage Act, 50 U. S. C., 31 and 32, as amended. Its transmission or the revelation of its contents in any manner to an unauthorized person is prohibited by law."

"The above Espionage Notice can be disregarded unless this document is plainly marked with a security classification as "Restricted," "Confidential," "Secret," or "Top Secret."

---

The U. S. Government is absolved from any litigation which may ensue from the contractor's infringing on the foreign patent rights which may be involved.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE JULY 1949		3. REPORT TYPE AND DATES COVERED FINAL JANUARY - JULY 1949
4. TITLE AND SUBTITLE TRANSONIC FLOW THROUGH NEARLY CIRCULAR DUCTS WITH GRADUALLY VARYING CROSS-SECTION			5. FUNDING NUMBERS	
6. AUTHOR(S) DON BLOSE				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) AIRCRAFT LABORATORY ENGINEERING DIVISION (WIND TUNNEL BRANCH) AIR FORCE MATERIEL COMMAND WRIGHT-PATTERSON AFB, OH 45433			8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AIRCRAFT LABORATORY ENGINEERING DIVISION (WIND TUNNEL BRANCH) AIR FORCE MATERIEL COMMAND WRIGHT-PATTERSON AFB, OH 45433			10. SPONSORING/MONITORING AGENCY REPORT NUMBER  AF-TR-5790	
11. SUPPLEMENTARY NOTES				
12a. DISTRIBUTION AVAILABILITY STATEMENT APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) A METHOD FOR THE ANALYSIS OF A TRANSONIC FLOW FIELD IN A NEARLY CIRCULAR DUCT WITH GRADUAL CHANGES IN CROSS-SECTION IS DEVELOPED. A LINEARIZED DIFFERENTIAL EQUATIONF FOR THE DEVIATION FROM THE RESULTS OF THE ONE-DIMENSIONAL THEORY IS PRESENTED AND IS SOLVED BY THE SUPERPOSITION OF PARTICULAR SOLUTIONS OBTAINED BY A PRODUCT HYPOTHESIS. THE POTENTIAL EQUATION, IN THIS CASE, WAS SIMPLIFIED FOR THE VICINITY OF THE SONIC VELOCITY. APPLICATION TO A CIRCULAR SUBSONIC-SUPERSONIC DUCT IS MADE IN ORDER TO DETERMINE THE MAGNITUDE OF PROPAGATED SUBSONIC DISTURBANCES AND THE RESULTING VELOCITY DISTRIBUTION.				
14. SUBJECT TERMS TRANSONIC FLOW, SUBSONIC DUCT, SUPERSONIC DUCT			15. NUMBER OF PAGES 36	
			16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT  UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE  UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT  UNCLASSIFIED	20. LIMITATION OF ABSTRACT  SAR	

AF TECHNICAL REPORT NO. 5790

TRANSONIC FLOW  
THROUGH NEARLY CIRCULAR DUCTS  
WITH GRADUALLY VARYING CROSS-SECTION

Don Blose

July 1949

Published by  
UNITED STATES AIR FORCE  
AIR MATERIEL COMMAND

Wright-Patterson Air Force Base, Dayton, Ohio

### ABSTRACT

A method for the analysis of a transonic flow field in a nearly circular duct with gradual changes in cross-section is developed. A linearized differential equation for the deviation from the results of the one-dimensional theory is presented and is solved by the superposition of particular solutions obtained by a product hypothesis. The potential equation, in this case, was simplified for the vicinity of the sonic velocity.

Application to a circular subsonic-supersonic duct is made in order to determine the magnitude of propagated subsonic disturbances and the resulting velocity distribution.

### PUBLICATION APPROVAL

For the Commanding General:

*Ernest M. Ljunggren Lt. Col. USAF*  
O. J. RITLAND  
Colonel, USAF  
Chief, Aircraft Laboratory  
Engineering Division  
Air Materiel Command

## FOREWORD

The author is greatly indebted to Dr. K. Gottfried Guderley for the initial idea of this transonic flow analysis method and for his valuable assistance during its development.

This project was initiated at the direction of the Wind Tunnel Branch, Aircraft Laboratory, Engineering Division, Air Materiel Command, and was administered under Expenditure Order No. 903-1428.

# TABLE OF CONTENTS

	Page
Section I - Introduction . . . . .	1
Section II - Theoretical Considerations . . . . .	2
Simplification of the Flow Differential Equation Near the Sonic Speed . . . . .	2
Pressure and Density Relations . . . . .	2
The Boundary Conditions . . . . .	3
The Basic Hypothesis for the Solution . . . . .	3
The Method of Solution . . . . .	7
Conditions for the Solution of Equations (1.33) and (1.34) . .	8
Section III - Applications . . . . .	8
Flow Through A Circular Subsonic-Supersonic Duct . . . . .	8
Determination of the Constants . . . . .	12
Discussion of Results . . . . .	13
A Remark on the Application to a Non-Symmetric Configuration .	14
Solution with $\Phi_{Bx}$ as A Function of $x^*$ . . . . .	15
Concluding Remarks . . . . .	15
Section IV . . . . .	16
Appendix A - Derivation of the Potential Equation . . . . .	16
Appendix B - Transonic Law of Similarity . . . . .	18
Appendix C - Boundary Condition . . . . .	20
Appendix D - Derivation of the Basic Flow Velocity . . . . .	21
Appendix E - Analytical Representation of the Constants $I_{0\mu}$ , $K_{0\mu}$ , $I_{v\mu}$ , and $K_{v\mu}$ . . . . .	21
Appendix F - Solution of Equation (1.33) if $\Phi_{Bx}$ is Given by $x^*$ . . . . .	24

## LIST OF SYMBOLS

$a$	velocity of sound
$a_{\nu\mu}^*$	functions of $x$ fulfilling differential equation for $\Phi$
$r$	radius of the duct
$x, y, \omega$	cylindrical coordinates
$Z_{\nu\mu}$	functions of $y$ fulfilling differential equation for $\Phi$
$\gamma$	ratio of specific heats, for air $\gamma = 1.4$
$\Phi$	total velocity potential
$\phi$	perturbation velocity potential
$\Phi_s$	one-dimensional flow potential
$\bar{\Phi}$	deviation from one-dimensional flow potential

## SUBSCRIPTS

$x, y, \omega$	refer to partial derivatives with respect to the cylindrical coordinates
$o$	refers to symmetric duct configuration
$u$	refers to unsymmetric duct configuration

## SUPERSSCRIPTS

$*$	refers to conditions at the duct throat
	refers to partial derivatives with respect to the $x$ coordinate



## SECTION I

### INTRODUCTION

This report presents an analytical method for the calculation of the flow field in a nearly circular duct with relatively gradual changes in cross-section. Due to simplifications introduced by means of the transonic law of similarity, the method in its present form would be restricted to flow fields with Mach numbers close to unity.

The investigation was initiated in connection with a plan for modifying the 10-Foot Wind Tunnel at Wright-Patterson Air Force Base. It was intended to vary the cross-sectional area with small inserts, rather than by a variation of the radius. A method was evolved to obtain the magnitude of deviations from the ideal parallel flow which might be caused by such an insert design. Before the actual computations were made the design was modified. However, the method itself is of interest, in that there are many other applications. An application, referring mainly to a modified test section design, will be given in the second part of this report.

The present analysis begins with a basic flow as given by one-dimensional theory. Any deviations from the basic flow are considered small, so that they can be determined by a linearized partial differential equation. The general solution to the differential equation is expressed by the superposition of particular solutions, each of which is the product of a Bessel function dependent on the radial distance  $y$  from the duct axis, a trigonometric function of the angular position  $\omega$ , and a function of the axial position,  $x$ . Of special interest is the last function which is influenced by the duct shape.

If one is interested only in supersonic flow analysis, an alternate method of computation is the well-known method of characteristics. The method of characteristics is more general, in that the condition of a slow change in cross-section is not required. The method of characteristics for three dimensions is similar to the present method in the requirement of a nearly circular cross-section. The major advantages of the present method are that the subsonic influences can be determined and that being essentially more analytical, a better picture of the flow behavior in general cases may be obtained. In some cases, depending upon the duct shape, the solutions can be obtained in a less tedious manner.

Briefly, the report starts with the simplification of the general flow differential equation by means of the transonic law of similarity. Then a general method of solution is developed, and finally this result is applied to some specific example.

## SECTION II

### THEORETICAL CONSIDERATIONS

#### Simplification of the Flow Differential Equation Near the Sonic Speed

This analysis is based on a cylindrical coordinate system (Figure 1). Let the  $\bar{x}$  axis be the duct center line,  $\bar{y}$ , the radial distance from the  $\bar{x}$  axis, and  $\omega$ , the angular position measured clockwise from the vertical position. The velocity potential is denoted by  $\Phi$ . Partial derivatives are denoted by subscripts (i.e.,  $\frac{\partial \Phi}{\partial \bar{x}} = \Phi_{\bar{x}}$ )

The general three-dimensional equation for a steady potential flow (derived in Appendix A) is given by

$$(1.1) \quad \Phi_{\bar{x}\bar{x}} \left(1 - \frac{\Phi_{\bar{x}}^2}{a^2}\right) - 2 \Phi_{\bar{x}\bar{y}} \left(\frac{\Phi_{\bar{x}} \Phi_{\bar{y}}}{a^2}\right) + \Phi_{\bar{y}\bar{y}} \left(1 - \frac{\Phi_{\bar{y}}^2}{a^2}\right) + \frac{\Phi_{\bar{y}}}{\bar{y}} + \frac{\Phi_{\omega\omega}}{\bar{y}^2} = 0$$

If deviations from a parallel sonic flow are small, the simplified transonic equation results. (Derivation may be found in Appendix B).

$$(1.2) \quad -(\gamma+1) \Phi_{xx} \Phi_x + \Phi_{yy} + \frac{\Phi_y}{y} + \frac{\Phi_{\omega\omega}}{y^2} = 0$$

where  $x = \frac{\bar{x}}{r_0}$

$y = \frac{\bar{y}}{r_0}$

$$(1.3) \quad \Phi(x, y) = \frac{1}{r_0 a^*} (\phi - \alpha^* \bar{x})$$

and  $r_0$  is the radius at the throat.

#### The Pressure and Density Relations

The new potential term  $\Phi(x, y)$  is directly related to the velocity.

However, one may be interested in the pressure and density relations at various positions along the duct. Bernoulli's law for compressible non-viscous gases is given by

$$(1.4) \quad \frac{\Delta p}{\rho} + u \Delta u + v \Delta v = 0$$

The last term is negligible, based on earlier assumptions.

Because  $\Delta u = \Phi_x \alpha^*$ , one can write

$$\Delta p = -\rho^* \alpha^* (\Phi_x \alpha^*)$$

Since  $\gamma \frac{p^*}{\rho^*} = a^{*2}$ , the preceding equation is rewritten as

$$(1.5) \quad \frac{\Delta p}{p^*} = -\gamma \varphi_x$$

### The Boundary Conditions

Let  $\bar{r}$  be the value of  $y$  at a point on the duct surface so that  $\bar{r} = \bar{r}(x, \omega)$ . This function, in the case of a vertical line of symmetry is expressed as

$$(1.6) \quad \bar{r} = \bar{r}_0(x) + \sum_1^{\infty} (\bar{r}_n(x) \cos n\omega)$$

The functions  $r_0, r_1, r_2, \dots, r_n$  are obtained by a Fourier analysis at the specific  $x$  positions, and are written as

$$(1.6a) \quad \bar{r}_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \bar{r}(x, \omega) d\omega$$

$$(1.6b) \quad \bar{r}_n(x) = \int_0^{2\pi} \bar{r}(x, \omega) \cos n\omega d\omega$$

$$\text{Let } r = \frac{\bar{r}}{r_0}.$$

The boundary condition of zero flow normal to the duct surface must be fulfilled. If the differences from the sonic velocity are small, this condition is expressed as

$$(1.7) \quad -\frac{dr}{dx} + \varphi_y = 0$$

(The derivation may be found in Appendix C).

Let us introduce for the slope

$$(1.8) \quad \Theta = \frac{dr}{dx}$$

Then one has the relation

$$(1.9) \quad \varphi_y = \Theta_0 + \sum \Theta_v \cos v\omega$$

Where  $\Theta_v$  is an additional slope for the unsymmetrical configurations.

### The Basic Hypothesis for the Solution

The solution is based on one-dimensional flow theory, in which the velocity distribution over each cross section is considered constant. The cross section is written as functions of the distance from the

minimum cross-section. The potential of this basic flow is denoted by  $\Phi_B$

, where  $\frac{\partial \Phi_B}{\partial x} = \Phi_{Bx} = f(r, r_0)$ . Specifically, as derived in Appendix D, this gives

$$(1.10) \quad \Phi_{Bx} = \left( \frac{4}{\gamma+1} \frac{\Delta r}{r_0} \right)^{\frac{1}{2}}$$

The entire potential is expressed by

$$(1.11) \quad \Phi = \Phi_B(x) + \bar{\Phi}(x, \gamma, \omega)$$

The  $\bar{\Phi}$  represents the deviations from the basic flow, and shall be considered small enough that higher order terms can be neglected. If the values of the complete potential (Equation (1.11)) is inserted into the transonic differential equation (Equation (1.2)) and second order terms of  $\bar{\Phi}$  are neglected, the following inhomogeneous equation is given:

$$(1.12) \quad -(\gamma+1)(\Phi_{Bx} \bar{\Phi}_{xx} + \Phi_{Bxx} \bar{\Phi}_x) + (\bar{\Phi}_{\gamma\gamma} + \frac{\bar{\Phi}_\gamma}{\gamma} + \frac{\bar{\Phi}_{\omega\omega}}{\gamma^2}) = (\gamma+1) \Phi_{Bx} \Phi_{Bxx}$$

For the representation of  $\bar{\Phi}$ , let us introduce the following functions. Let a Bessel function of  $\nu$ th order and of the argument  $\eta$  be denoted by  $J_\nu(\eta)$ . Let  $\bar{\eta}_{\nu\mu}$  be the  $\mu$ th argument (where  $\mu = 1, 2, 3 \dots$ ) of the Bessel function  $J_\nu$ , for which  $\frac{dJ_\nu(\eta)}{d\eta} = 0$ . The functions  $Z_{\nu\mu}$  are given by

$$(1.13) \quad Z_{\nu\mu}(\gamma) = J_\nu(\gamma \bar{\eta}_{\nu\mu})$$

The functions  $Z_{\nu\mu}$  fulfill the following differential equation

$$(1.14) \quad \frac{d^2}{d\gamma^2}(Z_{\nu\mu}) + \frac{d}{d\gamma} \left( \frac{Z_{\nu\mu}}{\gamma} \right) + (\bar{\eta}_{\nu\mu}^2 - \frac{\nu^2}{\gamma^2}) Z_{\nu\mu} = 0$$

There exists a relationship of orthogonality between the  $Z_{\nu\mu}$ , so that for  $\mu \neq \nu$

$$(1.15) \quad \int_0^1 \gamma Z_{\nu\mu} Z_{\nu\mu} d\gamma = 0$$

Let us introduce

$$(1.16) \quad \int_0^1 \gamma Z_{\nu\mu}^2 d\gamma = K_{\nu\mu}$$

A hypothesis for the perturbation velocity potential is chosen in such a way that the boundary conditions are fulfilled.

$$(1.17) \quad \Phi = \frac{1}{2} \gamma^2 \Theta_0(x) + \sum_{\nu=1}^{\infty} \left( \frac{1}{\nu} \gamma^{\nu} \Theta_{\nu}(x) \cos \nu \omega \right) + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{\nu\mu}(x) Z_{\nu\mu}(\gamma) \cos \nu \omega)$$

The functions  $a_{0\mu}(x)$  and  $a_{\nu\mu}(x)$  ( $\nu = 1, 2, 3, \dots$ ) must be determined in such a way that the differential equations for  $\Phi$  (Equation (1.12) is fulfilled. Inserting this hypothesis into Equation (1.12), one obtains

$$(1.18) \quad \begin{aligned} & -(\gamma+1) \left[ \Phi_{Bx} \sum_{\mu=1}^{\infty} (a_{0\mu}'' Z_{0\mu}) + \Phi_{Bxx} \sum_{\mu=1}^{\infty} (a_{0\mu}' Z_{0\mu}) \right] \\ & + \sum_{\mu=1}^{\infty} a_{0\mu} \left( \frac{d^2 Z_{0\mu}}{d\gamma^2} + \frac{1}{\gamma} \frac{dZ_{0\mu}}{d\gamma} \right) - (\gamma+1) \left[ \frac{1}{2} \gamma^2 (\Phi_{Bx} \Theta_0'' + \Phi_{Bxx} \Theta_0') \right] \\ & + 2 \Theta_0 - (\gamma+1) \left[ \Phi_{Bx} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{\nu\mu}'' Z_{\nu\mu} \cos \nu \omega) + \Phi_{Bxx} \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} (a_{\nu\mu}' Z_{\nu\mu} \cos \nu \omega) \right] \\ & + \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} a_{\nu\mu} \left( \frac{d^2 Z_{\nu\mu}}{d\gamma^2} \cos \nu \omega + \frac{dZ_{\nu\mu}}{d\gamma} \frac{\cos \nu \omega}{\gamma} - \frac{\gamma^2}{\gamma^2} Z_{\nu\mu} \cos \nu \omega \right) - (\gamma+1) \left[ \sum_{\nu=1}^{\infty} \frac{1}{\nu} \gamma^{\nu} \cos \nu \omega (\Phi_{Bx} \Theta_{\nu}'' + \Phi_{Bxx} \Theta_{\nu}') \right] = -(\gamma+1) (\Phi_{Bx} \Phi_{Bxx}) \end{aligned}$$

All the terms of this equation which contain trigonometric terms of (where  $\nu = 1, 2, 3, 4$ , etc.) must vanish separately.

For  $\nu = 0$  one has the condition

$$(1.19) \quad \begin{aligned} & -(\gamma+1) \left[ \Phi_{Bx} \sum_{\mu=1}^{\infty} (a_{0\mu}'' Z_{0\mu}) + \Phi_{Bxx} \sum_{\mu=1}^{\infty} (a_{0\mu}' Z_{0\mu}) \right] \\ & + \sum_{\mu=1}^{\infty} a_{0\mu} \left( \frac{d^2 Z_{0\mu}}{d\gamma^2} + \frac{1}{\gamma} \frac{dZ_{0\mu}}{d\gamma} \right) - (\gamma+1) \left[ \frac{\gamma^2}{2} (\Phi_{Bx} \Theta_0'' + \Phi_{Bxx} \Theta_0') \right] \\ & = -2 \Theta_0 - (\gamma+1) \Phi_{Bx} \Phi_{Bxx} \end{aligned}$$

The conditions for  $\nu = 1, 2, 3, \dots$  are

$$(1.20) \quad \begin{aligned} & -(\gamma+1) \left[ \Phi_{Bx} \sum_{\mu=1}^{\infty} (a_{\nu\mu}'' Z_{\nu\mu}) + \Phi_{Bxx} \sum_{\mu=1}^{\infty} (a_{\nu\mu}' Z_{\nu\mu}) \right] \\ & + \sum_{\mu=1}^{\infty} a_{\nu\mu} \left( \frac{d^2 Z_{\nu\mu}}{d\gamma^2} + \frac{1}{\gamma} \frac{dZ_{\nu\mu}}{d\gamma} - \frac{\gamma^2}{\gamma^2} Z_{\nu\mu} \right) \\ & = \left( \frac{\gamma+1}{\nu} \right) \gamma^{\nu} (\Phi_{Bx} \Theta_{\nu}'' + \Phi_{Bxx} \Theta_{\nu}') \end{aligned}$$

It can readily be seen from Equation (1.10) that

$$(1.21) \quad 2 \Theta_0 = -(\gamma+1) \Phi_{Bx} \Phi_{Bxx}$$

Further simplifications are made using Equation (1.14) with Equations (1.19) and (1.20), respectively.

$$(1.22) \quad -(\gamma+1) \sum_{\mu=1}^{\infty} [Z_{0\mu} (\Phi_{Bx} a_{0\mu}'' + \Phi_{Bxx} a_{0\mu}') ] \\ - \sum_{\mu=1}^{\infty} [\bar{\eta}_{0\mu}^2 Z_{0\mu} a_{0\mu}] = \frac{(\gamma+1)}{2} \gamma^2 (\Phi_{Bx} \Theta_0'' + \Phi_{Bxx} \Theta_0')$$

$$(1.23) \quad -(\gamma+1) \sum_{\mu=1}^{\infty} [Z_{v\mu} (\Phi_{Bx} a_{v\mu}'' + \Phi_{Bxx} a_{v\mu}') ] \\ - \sum_{\mu=1}^{\infty} [\bar{\eta}_{v\mu}^2 Z_{v\mu} a_{v\mu}] = \frac{(\gamma+1)}{\gamma} \gamma^v (\Phi_{Bx} \Theta_v'' + \Phi_{Bxx} \Theta_v')$$

The equations for the perturbation functions  $a_{v\mu}$  are obtained by multiplying with the term  $\gamma Z_{v\mu}$  and integrating between  $y = 0$  to  $y = 1$ . Because of the orthogonality relations for the Bessel functions, all with the terms  $\int \gamma Z_{v\mu} Z_{v\mu} dy$  drop out except for  $\mu = v$ . Inserting these relations in Equations (1.22) and (1.23) one obtains Equations (1.24) and (1.25), respectively.

$$(1.24) \quad I_{0\mu} [\Phi_{Bx} a_{0\mu}'' + \Phi_{Bxx} a_{0\mu}' + \frac{\bar{\eta}_{0\mu}^2}{\gamma+1} a_{0\mu}] \\ = K_{0\mu} \frac{1}{2} (\Phi_{Bx} \Theta_0'' + \Phi_{Bxx} \Theta_0')$$

where

$$(1.25) \quad K_{0\mu} = \int_0^1 Z_{0\mu}(\gamma) \gamma^3 dy = \frac{2}{\bar{\eta}_{0\mu}^2} J_0(\gamma \bar{\eta}_{0\mu})$$

$$(1.26) \quad I_{0\mu} = \int_0^1 Z_{0\mu}(\gamma) \gamma dy = \frac{1}{2} J_0^2(\gamma \bar{\eta}_{0\mu})$$

(Derivation may be found in the Appendix E). Also,

$$(1.27) \quad I_{v\mu} [\Phi_{Bx} a_{v\mu}'' + \Phi_{Bxx} a_{v\mu}' + \frac{\bar{\eta}_{v\mu}^2}{\gamma+1} a_{v\mu}] \\ = -K_{v\mu} \frac{1}{\gamma} (\Phi_{Bx} \Theta_v'' + \Phi_{Bxx} \Theta_v')$$

$$(1.28) \quad I_{v\mu} = \int_0^1 Z_{v\mu}(\gamma) \gamma dy = J_v^2(\bar{\eta}_{v\mu} \gamma)$$

$$(1.29) \quad K_{v\mu} = \int_0^1 Z_{v\mu}(\gamma) \gamma^{v+1} dy = \frac{\gamma}{\bar{\eta}_{v\mu}^2} J_v(\bar{\eta}_{v\mu} \gamma)$$

From Equations (1.24) and (1.27), and the boundary conditions which will be given later, the values of  $a_{v\mu}$  can be found. The resulting expression for the perturbation potential (Equation (1.17)) is written in a more convenient manner if one expresses the  $1/2 \gamma^2$  and  $\frac{1}{\gamma} \gamma^v$  by means of  $Z_{0\mu}$  and  $Z_{v\mu}$ .

$$\frac{1}{2} \gamma^2 = \sum b_{0\mu} Z_{0\mu}(\gamma)$$

$$\frac{1}{\gamma} \gamma^v = \sum b_{v\mu} Z_{v\mu}(\gamma)$$

where  $b_{0\mu} = \frac{\Theta_0}{2} \frac{K_{0\mu}}{I_{0\mu}}$

Let us introduce

$$(1.30) \quad b_{0\mu} + a_{0\mu} = a_{0\mu}^*$$

$$(1.31) \quad b_{v\mu} + a_{v\mu} = a_{v\mu}^*$$

The perturbation potential is then based entirely in terms of the functions  $z_{v\mu}$  so that

$$(1.32) \quad \bar{\Phi} = \sum_{\mu=0}^{\infty} [a_{0\mu}^* z_{0\mu}] + \sum_{\mu=0}^{\infty} \sum_{v=1}^{\infty} [a_{v\mu}^* z_{v\mu}]$$

One can then write the differential equations for  $a_{v\mu}^*$  by the introduction of Equations (1.30) and (1.31) into Equations (1.24) and (1.27).

$$(1.33) \quad \Phi_{Bx} a_{0\mu}^{*''} + \Phi_{Bxx} a_{0\mu}^{*'} + \frac{\bar{\eta}_{0\mu}^2}{\delta+1} a_{0\mu}^* = \frac{K_{0\mu}}{2 I_{0\mu}} \frac{\bar{\eta}_{0\mu}^2}{\delta+1} \Theta_0$$

$$(1.34) \quad \Phi_{Bx} a_{v\mu}^{*''} + \Phi_{Bxx} a_{v\mu}^{*'} + \frac{\bar{\eta}_{v\mu}^2}{\delta+1} a_{v\mu}^* = \frac{K_{v\mu}}{2 I_{v\mu}} \frac{\bar{\eta}_{v\mu}^2}{\delta+1} \Theta_v$$

$\Phi$  and its first derivatives with respect to  $x$  are continuous. Therefore, according to this relation,  $a_{v\mu}^*$  and their first derivatives are continuous.

One might ask why the hypothesis (Equation (1.32)) was not used from the beginning instead of Equation (1.17). The reason is that the second derivative with respect to  $y$  does not converge (because of the character of the function  $y^v$ ). Therefore, all of the considerations regarding the fulfillment of the differential equation for  $\Phi$  would have been meaningless.

#### Method of Solution

The solution of the perturbation functions  $a_{v\mu}^*$  for the differential equations (1.33) and (1.34) is found for any duct, within the defined limitations, by means of numerical integration. For suitable configurations, analytical solutions can be found. In some cases, a piecewise analytical solution may be satisfactory. In any case, the pressure distribution along the axis is influenced only by the expressions with  $v = 0$ , since the nature of higher order Bessel functions is such that there is no effect along the axis.

Deviation from a parallel flow are found from  $\Phi_y$ . The only expressions which are of importance along the axis are those for  $v = 0$  and  $v = 2$ . Actually, an infinite number of terms are needed to describe the flow completely, but the higher terms in  $\mu$  and  $v$  are very small and are of influence only close to the wall.

### Conditions for the Solution of Equations (1.33) and (1.34)

Equations (1.33) and (1.34) are of the second order, and consequently the solutions must be determined by two boundary conditions. The first condition is obvious. With the assumption of a subsonic duct extending to negative infinity, one may assume that the values of the perturbation terms  $a_{v\mu}^*$  should vanish as one approaches negative infinity.

The character of the second condition is less obvious. One may first ask at which point such a condition might be given. Certainly it cannot be a point in the supersonic region, for the fulfillment of the condition will affect the entire solution, including points upstream of the point where the boundary condition is given. Since this would violate the law of the forbidden signals, the location of the other condition should be expected to be in the subsonic region. Actually, mathematical reasoning leads to the location of the boundary condition at the narrowest cross-section. If one rewrites Equations (1.33) and (1.34) in such a way that the coefficient of the highest derivative of the term  $a_{v\mu}^*$  is one, the coefficients of the other terms become singular at the sonic velocity. The solution of the differential equation may be singular at any point where these coefficients are singular.

Consider now only the solutions of the homogeneous part of the differential equation, since the influence of change in the arbitrary constants and consequently, the influence of the boundary condition is limited to that part. Let the basic flow velocity be represented by  $\Phi_{0x} = x^{\bar{\alpha}}$ . Then the two linearly independent solutions of the homogeneous differential equation are represented by

$$(1.35) \quad a_{v\mu}^* = P(x)$$

$$(1.36) \quad a_{v\mu}^* = x'^{-\bar{\alpha}} P(x)$$

where  $P$  denotes a power series of  $x$ , which starts with the absolute term.

As the term  $\Phi_{0x}$  approaches zero (at the throat) we have  $\bar{\alpha} > 0$ . Then the expression  $a_{v\mu}^*$ , from Equation (1.36) will tend to infinity as  $x \rightarrow 0$ . Since this is physically impossible, the second condition is that remains finite at the throat.

To summarize the boundary conditions, the flow deviation expressions  $a_{v\mu}^*$  must be finite at  $x = 0$  and zero as  $x \rightarrow -\infty$ .

### SECTION III

#### APPLICATIONS

#### Flow Through A Circular Subsonic-Supersonic Duct

We apply this theory to an axially-symmetric transonic duct configuration. This duct is similar to the modified 10-Foot Wind Tunnel test section



at Wright-Patterson Air Force Base. The duct is first cylindrical (Section A, Figures 4 and 5) followed by a throat region of parabolic profile (Section B), and finally the duct expands conically (Section C). The configuration is geometrically determined by the subsonic junction position  $x_0$ , the radius of curvature  $R$ , and the angle of expansion  $\alpha$ . It is desired to trace the magnitudes of the x-velocity deviations from a uniform, parallel flow through the duct channel.

In the differential equation (1.33) for  $a_{0,\mu}^*$ , the shape of the duct expresses itself by the function  $\Phi_B(x)$ , which occurs in the coefficients of  $a_{0,\mu}^{*''}$  and  $a_{0,\mu}^{*'}$ , and furthermore, in the function  $\Theta_0$  which occurs in the righthand term.

In Section A, the duct is cylindrical and consequently the radius, slope, and  $\Phi_{Bx}$  terms are

$$(2.1) \quad r = r_0 = 1 + \frac{x_0^2 r_0}{2R}$$

(the value of  $r_0$  can be seen from the subsequent formula (2.4)).

$$(2.2) \quad \Theta_0 = 0$$

$$(2.3) \quad \Phi_{Bx} = \left( \frac{4}{\gamma+1} C \right)^{\frac{1}{2}} \quad (\text{for section A})$$

The profile of Section B is an approximation for a circular arc with the radius of curvature  $R$ . When approximated by a parabola with the vertex at the minimum section of the duct, the shape is expressed by:

$$(2.4) \quad r = 1 + \frac{x^2 r_0}{2R}$$

$$(2.5) \quad \Theta_0 = \frac{x r_0}{R}$$

$$(2.6) \quad \Phi_{Bx} = x \left( \frac{2}{\gamma+1} \frac{r_0}{R} \right)^{\frac{1}{2}} \quad (\text{for section B})$$

Section C starts tangential to the parabolic Section B and expands conically at the angle  $\alpha$ . The shape terms are

$$(2.7) \quad r = 1 + \left( x\alpha - \frac{R\alpha^2}{2r_0} \right)$$

$$(2.8) \quad \Theta_0 = \alpha$$

$$(2.9) \quad \Phi_{Bx} = \left( \frac{4\alpha}{\gamma+1} \left( x - \frac{R\alpha}{2r_0} \right) \right) \quad (\text{for section C})$$

These values inserted into the differential equation (1.28) lead to the specific differential equation for each section.

Section A

$$(2.10) \quad \Phi_{Bx} a_{0,\mu}^{*''} + \frac{\eta_{0,\mu}^2}{\gamma+1} a_{0,\mu}^* = 0$$

### Section B

$$(2.11) \quad x a_{o\mu}^{*''} + a_{o\mu}^{*'} + E_{1\mu} a_{o\mu}^* = E_{2\mu} x$$

The constants are:

$$(2.12) \quad E_{1\mu} = \frac{\frac{\bar{\eta}_{o\mu}^2}{\delta+1}}{\left(\frac{2}{\delta+1} \frac{r_o}{R}\right)^{\frac{1}{2}}}$$

$$(2.13) \quad E_{2\mu} = \frac{r_o}{2R} \frac{\frac{\bar{\eta}_{o\mu}^2}{\delta+1} \frac{K_{o\mu}}{I_{o\mu}}}{\left(\frac{2}{\delta+1} \frac{r_o}{R}\right)^{\frac{1}{2}}}$$

### Section C

$$(2.14) \quad \left[ \frac{4\alpha}{\delta+1} \left( x - \frac{R\alpha}{2r_o} \right) \right]^{\frac{1}{2}} a_{o\mu}^{*''} + \frac{1}{2} \left( \frac{4\alpha}{\delta+1} \right)^{\frac{1}{2}} \left( x - \frac{R\alpha}{2r_o} \right)^{-\frac{1}{2}} a_{o\mu}^{*'} + \frac{\bar{\eta}_{o\mu}^2}{\delta+1} a_{o\mu}^* = \frac{K_{o\mu}}{2I_{o\mu}} \frac{\bar{\eta}_{o\mu}}{\delta+1} \alpha$$

The solutions of the differential equations (2.10), (2.11), and (2.14) will now be determined. The complete solution, in each case, is the sum of the general solution of the homogeneous portion and a particular solution of the non-homogeneous equation. The constants will be determined later.

### Section A

The solution of Equation (2.10) is

$$(2.15) \quad a_{o\mu}^* = C_{1\mu} e^{\left(-\frac{\bar{\eta}_{o\mu}^2}{\Phi_{Bx}(\delta+1)}\right)^{\frac{1}{2}} x} + C_{2\mu} e^{-\left(-\frac{\bar{\eta}_{o\mu}^2}{\Phi_{Bx}(\delta+1)}\right)^{\frac{1}{2}} x}$$

### Section B

The particular solution of Equation (2.11) is

$$(2.16) \quad a_{o\mu}^* = E_{3\mu} + E_{4\mu} x$$

where

$$(2.17) \quad E_{3\mu} = -\frac{r_o}{2R} \frac{K_{o\mu}}{I_{o\mu}} \frac{\left(\frac{2}{\delta+1} \frac{r_o}{R}\right)^{\frac{1}{2}}}{\frac{\bar{\eta}_{o\mu}^2}{\delta+1}}$$

$$(2.18) \quad E_{4\mu} = \frac{r_o}{2R} \frac{K_{o\mu}}{I_{o\mu}}$$

The homogeneous part of Equation (2.11), through a simple transformation, is of the form

$$(2.19) \quad y'' + \frac{1}{x} y' - \left[ \frac{1}{x} + \left(\frac{p}{2x}\right)^2 \right] y = 0$$

the solution<sup>1</sup> of which is

$$(2.20) \quad \gamma = Z_p \left[ 2 \lambda x^{\frac{1}{2}} \right]$$

The term  $Z_p$  denotes any linear combination of Bessel functions of order  $p$ . The general solution of the homogeneous part of Equation (2.11) then is such that the complete solution becomes

$$(2.21) \quad \alpha_{0,\mu}^* = E_{3,\mu} + E_{4,\mu} x + C_{3,\mu} J_0 \left[ 2 E_{1,\mu}^{\frac{1}{2}} x^{\frac{1}{2}} \right] + C_{4,\mu} N_0 \left[ 2 E_{1,\mu}^{\frac{1}{2}} x^{\frac{1}{2}} \right]$$

### Section C

A particular solution of Equation (2.14) in Section C is given by

$$(2.22) \quad \alpha_{0,\mu}^* = \frac{K_{0,\mu}}{2 I_{0,\mu}} \alpha$$

In order to find the general solution of the homogeneous part of Equation (1.28), the transformations  $u = \left( x - \frac{R\alpha}{2r_0} \right)^{\frac{1}{2}}$  and  $\alpha_{0,\mu}^*(x) = q(u)$  are carried out. The differential equation is then written

$$(2.23) \quad \frac{1}{4u} \left( q'' - \frac{q}{u} \right) + \frac{1}{2u} \left( \frac{q}{2u} \right) + E_{5,\mu} q = 0$$

where

$$(2.24) \quad E_{5,\mu} = \frac{\bar{\eta}_{0,\mu}^2}{2[\alpha(\gamma+1)]^{\frac{1}{2}}}$$

Equation (2.23) has the form

$$(2.25) \quad \gamma + b x \gamma = 0$$

where the solution<sup>2</sup> is given as

$$(2.26) \quad \gamma = x^{\frac{1}{2}} Z_{\frac{1}{3}} \left[ \frac{1}{3} b^{\frac{1}{2}} x^{\frac{3}{2}} \right]$$

Consequently, the solution of Equation (2.23) is

$$(2.27) \quad q = u^{\frac{1}{2}} Z_{\frac{1}{3}} \left[ \frac{4}{3} E_{5,\mu}^{\frac{1}{2}} u^{\frac{3}{2}} \right]$$

<sup>1</sup> Jahnke, E. and Emde, F., Tables of Functions. Fourth Edition. Dover Publications, 1945, p. 147.

<sup>2</sup> Ibid., p. 147.

If one substitutes for  $u$  and  $g(u)$ , and writes the constants for the explicit Bessel function, one finally obtains

$$(2.28) \quad a_{0\mu}^* = \frac{K_{0\mu}}{2 I_{0\mu}} \alpha + \left( \chi - \frac{R\alpha}{2 r_0} \right)^{\frac{1}{4}} \left[ C_{5\mu} J_{\frac{1}{3}}(e_1) + C_{6\mu} J_{-\frac{1}{3}}(e_1) \right]$$

$$(2.29) \quad a_{0\mu}^{*'} = E_{5\mu}^{\frac{1}{2}} \left[ C_{5\mu} J_{-\frac{2}{3}}(e_1) - C_{6\mu} J_{\frac{2}{3}}(e_1) \right]$$

where

$$(2.30) \quad e_1 = \frac{4}{3} E_{5\mu}^{\frac{1}{2}} \left( \chi - \frac{R\alpha}{2 r_0} \right)^{\frac{3}{4}}$$

#### Determination of the Constants

The constants  $C_{1\mu}, C_{2\mu}, C_{3\mu}, \dots$  and  $C_{6\mu}$  which occur in the solution for the functions  $a_{0\mu}^*$ , are found by equating the solutions of  $a_{0\mu}^*$ , given by Equations (2.15), (2.21), and (2.28) and their first derivatives for the points of junction. Furthermore, there are the conditions that  $a_{0\mu}^*$  remain finite at the throat and that they vanish as  $x$  tends to  $-\infty$ .

The condition that  $a_{0\mu}^*$  tends to zero for  $x \rightarrow -\infty$ , according to Equation (2.15) yields immediately

$$(2.31) \quad C_{2\mu} = 0$$

The condition that  $a_{0\mu}^*$  is finite at the throat requires that the coefficient of  $N_0$  in Equation (2.21) is zero.

$$(2.32) \quad C_{4\mu} = 0$$

The condition that  $a_{0\mu}^*$  and  $a_{0\mu}^{*}$  are continuous at the junction of Sections A and B yields the value

$$(2.33) \quad C_{3\mu} = \frac{E_{4\mu} + \left( -\frac{E_{1\mu}}{\Phi_{Bx}} \right)^{\frac{1}{2}} (E_{3\mu} + E_{4\mu} x_0)}{\left( -\frac{E_{1\mu}}{\Phi_{Bx}} \right)^{\frac{1}{2}} J_0 \left[ 2 E_{1\mu}^{\frac{1}{2}} x_0^{\frac{1}{2}} \right] + \frac{E_{1\mu}^{\frac{1}{2}}}{x_0^{\frac{1}{2}}} J_1 \left[ 2 E_{1\mu}^{\frac{1}{2}} x_0^{\frac{1}{2}} \right]}$$

The condition that  $a_{0\mu}^*$  and  $a_{0\mu}^{*}$  are continuous at the junction of Sections B and C yields the values

$$(2.34) \quad C_{5\mu} = \frac{E_{5\mu}^{\frac{1}{2}} J_{-\frac{1}{3}}(e_1) \left[ E_{3\mu} + E_{4\mu} x_1 + C_{3\mu} J_0(e_0) - \frac{K_{0\mu}}{2 I_{0\mu}} \alpha \right] + \left( \chi_1 - \frac{R\alpha}{2 r_0} \right)^{\frac{1}{4}} J_{-\frac{1}{3}}(e_1) \left[ E_{4\mu} - J_1(e_0) \frac{C_{3\mu} E_{1\mu}^{\frac{1}{2}}}{x_1^{\frac{1}{2}}} \right]}{\left( \chi_1 - \frac{R\alpha}{2 r_0} \right)^{\frac{1}{4}} E_{5\mu}^{\frac{1}{2}} \left[ J_{\frac{1}{3}}(e_1) J_{\frac{2}{3}}(e_1) + J_{-\frac{1}{3}}(e_1) J_{-\frac{2}{3}}(e_1) \right]}$$

$$(2.35) \quad C_{e\mu} = \frac{E_{s\mu}^{\frac{1}{2}} J_{-\frac{1}{2}}(e_0) \left[ E_{3\mu} + E_{4\mu} x_1 + C_{3\mu} J_0(e_0) - \frac{\kappa_{0\mu}}{2 I_{0\mu}} \alpha \right] - \left( x_1 - \frac{R\alpha}{2 r_0} \right)^{\frac{1}{2}} J_{\frac{1}{2}}(e_0) \left[ E_{4\mu} - J_1(e_0) \frac{C_{3\mu} E_{\mu}^{\frac{1}{2}}}{x_1} \right]}{\left( x_1 - \frac{R\alpha}{2 r_0} \right)^{\frac{1}{2}} E_{s\mu}^{\frac{1}{2}} \left[ J_{\frac{1}{2}}(e_0) J_{\frac{3}{2}}(e_0) + J_{-\frac{1}{2}}(e_0) J_{-\frac{3}{2}}(e_0) \right]}$$

where  $e_0 = 2 E_{\mu}^{\frac{1}{2}} x_0^{\frac{1}{2}}$

By the use of the previous formulae, the expressions  $\alpha_{0\mu}^*$  is determined completely. The quantities  $x_0$ ,  $R$ , and  $\alpha$  which determine the duct geometry are determined by the specific problem under consideration.

### Discussion of Results

In the previous formulae the quantities  $x_0$ ,  $R$ , and  $\alpha$  are written in the general form. This, of course, enables us to determine the solutions for different configurations.

In the following analysis the values which correspond to the Wright-Patterson Air Force Base 10-Foot Wind Tunnel will be inserted. Before this is done, the effect of the choice of the subsonic junction position  $x_0$  is studied. Most investigations of a supersonic flow are made disregarding any influences of the subsonic portion of the duct, since there is no theoretical basis for its consideration. It is of interest to determine some numerical values of the subsonic effects. Therefore, a flow pattern determination will be made of a configuration which corresponds to the supersonic part of the 10-Foot Wind Tunnel, and three different subsonic junction positions.

Accordingly, the duct shape gives

$$R = 50 r_0 \quad \alpha = .15 \text{ degrees}$$

$$x_0 = -0.10, -0.20, -0.40$$

The values of  $\bar{\eta}_{0\mu}$  are the Bessel function arguments for which  $\frac{d}{d\eta} [J_0(\eta)] = 0$  and are

$$\bar{\eta}_{0,1} = 3.832$$

$$\bar{\eta}_{0,2} = 7.015$$

$$\bar{\eta}_{0,3} = 10.1735$$

The values of  $E_{1\mu}$ ,  $E_{2\mu}$ ,  $E_{3\mu}$ , and  $E_{4\mu}$  are determined by Equations (2.12), (2.13), (2.17), and (2.18), respectively. The junction constants  $C_{3\mu}$ ,  $C_{5\mu}$ , and  $C_{6\mu}$  are obtained by means of Equations (2.33), (2.34), and (2.35), respectively.

Of special interest is the velocity distribution along the axis of the duct. The x-velocity terms found from one-dimensional theory are given by Equations (2.3), (2.6) and (2.9). The additional x-velocity along the axis caused by the perturbation potential  $\bar{\Phi}_x$  are found from Equation (1.32). Since  $J_0(\infty) = 1$ , one obtains

$$(2.36) \quad \bar{\Phi}_x = \sum_{\mu} a_{0\mu}^{*'}$$

Figure 6 shows the  $a_{0\mu}^{*'}$  values through Section B for  $x_0 = -.01, -.02, -.04, -\infty$ . In spite of appreciable initial flow deviations at some distance upstream from the throat, the effects are hardly noticeable at the minimum cross section and in the supersonic region. A corresponding behavior is found for the terms  $a_{02}^{*'}$ ,  $a_{03}^{*'}$ , etc. Accordingly, even extreme changes in the subsonic region are without importance for the supersonic region.

Figure 8 shows the velocity distribution along the axis according to the one-dimensional theory, and a velocity which includes the corrections given by  $a_{01}^{*'}$ ,  $a_{02}^{*'}$ , and  $a_{03}^{*'}$ . Beyond  $x = 0.5$  the velocity distribution is rather smooth, showing the present configuration to be satisfactory. The maximum velocity deviations from the one-dimensional theory have a magnitude corresponding to a Mach number of about .004. The velocity deviation at about  $x = 0.4$  is rather large, but this is to be expected from the following reasoning. Point M (Figure 3) is considered to be the junction between Sections B and C. The effect of the change of shape at this point propagates to the axis along the Mach lines MN. Upstream of MN the flow, especially along the axis, corresponds to the parabolic surfaces. Consequently, any change in pressure distribution caused by the transition is felt only downstream of point B. The one-dimensional theory (Figure 8, curve  $\Phi_{ax}$ ) shows the influence of the junctions to be right at point A. The superposition of the  $\bar{\Phi}_x$  value shows the occurrence of the change at  $x = .38$ . The x-coordinate of Point N in Figure 3, computed by an exact relationship for the Mach angle, is  $x = .387$ . Figure 8, curve  $(\Phi_{ax} + \bar{\Phi}_x)$  shows what might be expected from a more general reasoning. The functions  $a_{01}^{*'}$ ,  $a_{02}^{*'}$ , and  $a_{03}^{*'}$  are given in Figure 7.

#### A Remark on the Application to a Non-Symmetric Configuration

Although no examples of the application for an unsymmetric flow are carried out, an idea of the procedure will be given. The shape of the duct must be expressed according to Equation (1.6). For instance, one must carry out a harmonic analysis of the function  $r(x, \theta)$  with respect to  $\theta$  for various values of  $x$ . This can be done by a numerical or analytical evaluation of the integrals (1.6a) and (1.6b), or by means of a harmonic analyzer. Then the values of  $\Theta_v$  are found from Equation (1.9). Following this preparation, the procedure can be carried out in the same way as for an axially symmetric duct.

### Solution with $\Phi_{sx}$ as A Function of $X^*$

A solution of the differential equation is determined for the coefficient  $\Phi_{sx}$  as a power of the longitudinal position. The solution and its derivation will be found in the Appendix (Section IV, Part F).

### Concluding Remarks

A method for the analysis of transonic flow through a nearly circular duct with only gradual changes in cross-section has been derived. One is finally lead to the solutions of ordinary linear differential equations. (Equations (1.33) and (1.34)). These solutions can be carried out numerically, or even analytically in many cases.

Application of this method was made to a configuration similar to the Wright-Patterson Air Force Base 10-Foot Wind Tunnel. A velocity distribution through the duct was determined and the disturbances caused by the variation in duct shapes were computed. The velocity profile for the configuration is satisfactory. The subsonic influence in this case is small enough that the usual consideration of no subsonic disturbances propagated into the supersonic region is justified.

## SECTION IV

### APPENDIX A

#### Derivation of the Potential Equation

It follows from the vorticity theorems that, with certain conditions for the initial flow which are fulfilled in most practical cases, there exists a velocity potential  $\Phi(x, y, z)$ , where the coordinates are shown in Figure 1. (That is, the velocity vector can be expressed as the gradient of a scalar function  $\Phi$ ). If one chooses an arbitrary function,  $\Phi(x, y, z)$  and determines the velocity components accordingly; i.e.,

$$(3.1) \quad V_x = \Phi_x$$

$$(3.2) \quad V_y = \Phi_y$$

$$(3.3) \quad V_z = \Phi_z$$

and furthermore, determines the pressure from Bernoulli's equation, then Euler's equations of motion are automatically fulfilled. Therefore, if a velocity potential exists, we have to check only the condition of continuity. The continuity equation is given by

$$(3.4) \quad \frac{\partial}{\partial x}(\rho V_x) + \frac{1}{y} \frac{\partial}{\partial y}(\rho y V_y) + \frac{1}{y} \frac{\partial}{\partial z}(\rho V_z) = 0$$

If one inserts Equations (3.1), (3.2), and (3.3) into Equation (3.4)

$$(3.5) \quad \frac{\partial}{\partial x}(\rho \Phi_x) + \frac{\partial}{\partial y}(\rho \Phi_y) + \frac{\partial}{\partial z}(\rho \Phi_z) = 0$$

The derivatives of  $\rho$  are found from Bernoulli's equation for three-dimensional flow

$$(3.6) \quad i + \frac{1}{2}(\Phi_x^2 + \Phi_y^2 + \frac{\Phi_z^2}{y^2}) = \text{constant}$$

where  $i$  denotes the enthalpy, so that

$$i = \frac{a^2}{\gamma + 1}$$

From the isentropic flow relations, since  $i = C_p T$  for perfect gases.

$$\frac{i}{i_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1}$$

Consequently, one writes

$$\frac{di}{i} = (\gamma - 1) \frac{d\rho}{\rho}$$

Or

$$(3.7) \quad di = a^2 \frac{d\rho}{\rho}$$



If Equation (3.6) is differentiated, and Equation (3.7) is inserted the resulting expression is

$$a^2 \frac{\partial \rho}{\partial \bar{x}} + \phi_{\bar{x}} d\phi_{\bar{x}} + \phi_{\bar{y}} d\phi_{\bar{y}} + \frac{\phi_{\omega}}{\bar{y}} d\left(\frac{\phi_{\omega}}{\bar{y}}\right) = 0$$

Or

$$(3.8) \quad d\rho = -\frac{\rho}{a^2} (\phi_{\bar{x}} d\phi_{\bar{x}} + \phi_{\bar{y}} d\phi_{\bar{y}} + \frac{\phi_{\omega}}{\bar{y}} d\left(\frac{\phi_{\omega}}{\bar{y}}\right)) = 0$$

Writing Equation (3.8) in detail, one obtains

$$(3.8a) \quad \frac{\partial \rho}{\partial \bar{x}} = -\frac{\rho}{a^2} (\phi_{\bar{x}} \phi_{\bar{x}\bar{x}} + \phi_{\bar{y}} \phi_{\bar{x}\bar{y}} + \frac{\phi_{\omega} \phi_{\bar{x}\omega}}{\bar{y}^2})$$

$$(3.8b) \quad \frac{\partial \rho}{\partial \bar{y}} = -\frac{\rho}{a^2} (\phi_{\bar{x}} \phi_{\bar{x}\bar{y}} + \phi_{\bar{y}} \phi_{\bar{y}\bar{y}} + \frac{\phi_{\omega}^2}{\bar{y}^3} + \frac{\phi_{\omega} \phi_{\omega\bar{y}}}{\bar{y}^2})$$

$$(3.8c) \quad \frac{\partial \rho}{\partial \omega} = -\frac{\rho}{a^2} (\phi_{\bar{x}} \phi_{\bar{x}\omega} + \phi_{\bar{y}} \phi_{\bar{y}\omega} + \frac{\phi_{\omega} \phi_{\omega\omega}}{\bar{y}^2})$$

Now, if the partial derivatives of  $\rho$  from Equations (3.8a), (3.8b), and (3.8c) are inserted in Equation (3.5), one obtains the expression

$$(3.9) \quad -\frac{\rho}{a^2} (\phi_{\bar{x}} \phi_{\bar{x}\bar{x}} + \phi_{\bar{y}} \phi_{\bar{x}\bar{y}} + \frac{\phi_{\omega} \phi_{\bar{x}\omega}}{\bar{y}^2}) \phi_{\bar{x}} + \rho \phi_{\bar{x}\bar{x}} \\ - \frac{\rho}{a^2} (\phi_{\bar{x}} \phi_{\bar{x}\bar{y}} + \phi_{\bar{y}} \phi_{\bar{y}\bar{y}} + \frac{\phi_{\omega}^2}{\bar{y}^3} + \frac{\phi_{\omega} \phi_{\omega\bar{y}}}{\bar{y}^2}) \phi_{\bar{y}} + \frac{\rho \phi_{\bar{y}}}{\bar{y}} \\ + \rho \phi_{\bar{y}\bar{y}} - \frac{\rho}{a^2} (\frac{\phi_{\bar{x}} \phi_{\bar{x}\omega}}{\bar{y}} + \frac{\phi_{\bar{y}} \phi_{\bar{y}\omega}}{\bar{y}} + \frac{\phi_{\omega} \phi_{\omega\omega}}{\bar{y}^3}) \frac{\phi_{\omega}}{\bar{y}} + \rho \frac{\phi_{\omega\omega}}{\bar{y}^2} = 0$$

If one divides Equation (3.9) by  $\rho$  and collects the terms, one obtains the final equation

$$(3.10) \quad \phi_{\bar{x}\bar{x}} (1 - \frac{\phi_{\bar{x}}^2}{a^2}) - 2 \phi_{\bar{x}\bar{y}} (\frac{\phi_{\bar{x}} \phi_{\bar{y}}}{a^2}) + \phi_{\bar{y}\bar{y}} (1 - \frac{\phi_{\bar{y}}^2}{a^2}) \\ + \phi_{\omega\omega} (1 - \frac{\phi_{\omega}^2}{\bar{y}^2 a^2}) - 2 \frac{\phi_{\bar{x}} \phi_{\omega} \phi_{\bar{x}\omega}}{\bar{y}^2 a^2} - 2 \frac{\phi_{\bar{y}} \phi_{\omega} \phi_{\bar{y}\omega}}{\bar{y}^2 a^2} + \frac{\phi_{\bar{y}}}{\bar{y}} - \frac{\phi_{\omega}^2 \phi_{\bar{y}}}{\bar{y}^3 a^2} = 0$$

This is the complete potential equation for three-dimensional flow.

## APPENDIX B

### Transonic Law of Similarity

The general three-dimensional equation for a steady potential flow is given by:

$$(3.11) \quad \Phi_{\bar{x}\bar{x}} \left(1 - \frac{\Phi_{\bar{x}}^2}{a^2}\right) - 2 \frac{\Phi_{\bar{x}\bar{y}}(\Phi_{\bar{x}}\Phi_{\bar{y}})}{a^2} + \Phi_{\bar{y}\bar{y}} \left(1 - \frac{\Phi_{\bar{y}}^2}{a^2}\right) + \Phi_{\omega\omega} \left(1 - \frac{\Phi_{\omega}^2}{\bar{y}^4 a^2}\right) - 2 \frac{\Phi_{\bar{x}}\Phi_{\omega}\Phi_{\bar{x}\omega}}{\bar{y}^2 a^2} - 2 \frac{\Phi_{\bar{y}}\Phi_{\omega}\Phi_{\bar{y}\omega}}{\bar{y}^2 a^2} + \frac{\Phi_{\bar{y}}}{\bar{y}} - \frac{\Phi_{\omega}^2\Phi_{\bar{y}}}{\bar{y}^3 a^2} = 0$$

The velocity of sound  $a$  is obtained from Bernoulli's equation.

$$(3.12) \quad a^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} w^2$$

Where the total stream velocity:

$$w = \left( \Phi_{\bar{x}}^2 + \Phi_{\bar{y}}^2 + \frac{\Phi_{\omega}^2}{\bar{y}^2} \right)^{\frac{1}{2}}$$

Let us consider a family of flow patterns, where each flow pattern is characterized by a specific value of a parameter  $z$ . Let the velocity potential be given by:

$$(3.13) \quad \Phi = a^{*2} (\bar{x} + z \Phi(\bar{x}, \bar{y}, \omega) + z^2 \Phi(\bar{x}, \bar{y}, \omega) + \dots)$$

where

$$(3.14) \quad \tilde{x} = \bar{x}$$

$$(3.15) \quad \tilde{y} = \bar{y} z^{\frac{1}{2}}$$

This hypothesis is justified, by the fact that its introduction into the differential equation for  $\Phi$ , with the limiting process  $z \rightarrow 0$ , results into a non-trivial equation for  $\Phi$ . Indeed substitution of Equations (3.12), (3.13), (3.14), and (3.15) into the potential equation (Equation (3.11)) gives

$$(3.16) \quad a^{*2} z \Phi_{\bar{x}\bar{x}} \left(1 - \left(\frac{a^{*2} + a^{*2} z \Phi_{\bar{x}}}{a^2}\right)^2\right) - 2 a^{*2} z^{\frac{3}{2}} \Phi_{\bar{x}\bar{y}} \left(\frac{a^{*2} (1 + z \Phi_{\bar{x}}) z^{\frac{1}{2}} \Phi_{\bar{y}}}{a^2}\right) + a^{*2} z^2 \Phi_{\bar{y}\bar{y}} \left(1 - \frac{(a^{*2} z^{\frac{1}{2}} \Phi_{\bar{y}})^2}{a^2}\right) + \frac{a^{*2} z^2 \Phi_{\bar{y}}}{\bar{y}} + \frac{a^{*2} z^2 \Phi_{\omega\omega}}{\bar{y}^2} = 0$$

The value of the sonic velocity as given in Equation (3.12) can be written in terms of  $a^{*2}$  and the velocity potential, and is finally expressed as

$$(3.17) \quad a^2 = a^{*2} - (\gamma-1)(z \Phi_{\bar{x}}) a^{*2}$$

The introduction of Equation (3.17) into Equation (3.16) leads to

$$(3.18) \quad \begin{aligned} & \bar{z}^2 \Phi_{\bar{x}\bar{x}} [-\Phi_{\bar{x}}(\gamma+1) - \bar{z} \Phi_{\bar{x}}^2] - 2 \bar{z}^3 \Phi_{\bar{x}\bar{y}} (1 + \bar{z} \Phi_{\bar{x}}) \Phi_{\bar{y}} \\ & + \bar{z}^2 \Phi_{\bar{y}\bar{y}} [1 - (\gamma-1) \bar{z} \Phi_{\bar{x}} - \bar{z}^3 \Phi_{\bar{y}}^2] + \frac{\bar{z}^2 \Phi_{\bar{y}}}{\bar{y}} [1 - (\gamma-1) \bar{z} \Phi_{\bar{x}}] \\ & + \bar{z}^2 \frac{\Phi_{\bar{y}\bar{y}}}{\bar{y}^2} [1 - (\gamma-1) \bar{z} \Phi_{\bar{x}}] = 0 \end{aligned}$$

Since  $\bar{z}$  is small, only the terms of the lowest power of  $\bar{z}$  prevail. The equation is divided by  $\bar{z}^2$ , and one obtains

$$(3.19) \quad -(\gamma+1) \Phi_{\bar{x}\bar{x}} \Phi_{\bar{x}} + \Phi_{\bar{y}\bar{y}} + \frac{\Phi_{\bar{y}}}{\bar{y}} + \frac{\Phi_{\omega\omega}}{\bar{y}^2} = 0$$

This equation shall be referred to as the simplified transonic potential equation.

Let us consider Equations (3.13), (3.14), and (3.15) for small values of

$$(3.20) \quad \Phi = \alpha^* (\bar{x} + \bar{z} \Phi(\bar{x}, \bar{y}, \omega))$$

The velocity components given by

$$(3.21) \quad \Phi_{\bar{x}} = \alpha^* + \alpha^* \bar{z} \Phi_{\bar{x}}(\bar{x}, \bar{y}, \omega)$$

$$(3.22) \quad \Phi_{\bar{y}} = \alpha^* \bar{z}^{\frac{3}{2}} \Phi_{\bar{y}}(\bar{x}, \bar{y}, \omega)$$

$$(3.23) \quad \frac{\Phi_{\omega}}{\bar{y}} = \alpha^* \bar{z}^{\frac{3}{2}} \frac{\Phi_{\omega}}{\bar{y}}(\bar{x}, \bar{y}, \omega)$$

Let us consider flow patterns which belong to different values of the parameter  $\bar{z}$ . In such flow patterns the points having the same  $x$  and  $y$  values are called corresponding points. From Equations (3.14) and (3.15) it is seen that at corresponding points, the values of  $\bar{x}$  are the same and the  $\bar{y}$  values vary as  $\bar{z}^{-\frac{1}{2}}$ . At corresponding points the deviation of the  $\bar{x}$  velocity is proportional to  $\bar{z}$ , the  $\bar{y}$  velocity and the  $\omega$  velocity are both proportional to  $\bar{z}^{\frac{3}{2}}$ . The significance of

$\bar{z}$  is found from the conditions at infinity. At infinity the deviation of the  $\bar{x}$  velocity from the sonic velocity is proportional to the difference of the free stream Mach number from one. Therefore,

$\bar{z}$  is also proportional to the difference between the stream Mach number and one.

For  $\bar{z} = 0$  the streamlines have a constant  $\bar{y}$ . The deviations of the streamlines from a line ( $\bar{y} = \text{constant}$ ) are found by integrating the slope of the streamlines. The component of the velocity deviations of  $\omega$ , as well as the deviation of  $\bar{y}$ , behaves proportional to  $\bar{z}^{\frac{3}{2}}$ . Consequently, the deviation of the streamlines from the lines  $\bar{y} = \text{constant}$ ,  $\omega = \text{constant}$ , behave like  $\bar{z}^{\frac{3}{2}}$ . To summarize the transonic law of similarity:

1. The  $\bar{x}$  coordinates of corresponding points are the same
2. The  $y$  coordinates of corresponding points are proportional to  $\bar{z}^{-\frac{1}{2}}$
3. The difference between the  $x$  velocity and the sonic velocity is proportional to  $\bar{z}$
4. The difference between the pressure and the sonic pressure is proportional to  $\bar{z}$
5. The deviation of the streamlines from lines of constant  $y$  and  $\omega$  is proportional to  $\bar{z}^{\frac{1}{2}}$

## APPENDIX C

### Boundary Condition

Along a body at rest the velocity vector is parallel to the surface; that is, the component of flow normal to the surface is zero. To express the relation, let the unit vectors in the direction of increasing  $x$ ,  $y$  and  $\omega$  coordinates be  $U_x$ ,  $U_y$ , and  $U_\omega$ , respectively. The total velocity vector then is expressed by

$$(3.24) \quad a^* [(1 + \phi_x) U_x + \phi_y U_y + \frac{\phi_\omega}{r} U_\omega]$$

The vector component normal to the duct surface is given by

$$(3.25) \quad -\frac{\partial r}{\partial x} U_x - \frac{\partial r}{\partial \omega} U_\omega + U_y = 0$$

The scalar product of these two terms is the component of the velocity vector normal to the surface. According to the boundary condition, this component is zero. Then one obtains

$$(3.26) \quad -a^* (1 + \phi_x) \frac{\partial r}{\partial x} + a^* \phi_y - a^* \frac{\phi_\omega}{r^2} \frac{\partial r}{\partial \omega} = 0$$

The assumption that  $\phi$  is a small quantity implies that the slope of the duct and surface is gradual, and consequently  $\frac{\partial r}{\partial x}$  and  $\frac{\partial r}{\partial \omega}$  are small quantities.

Therefore, the expressions  $\phi_x \frac{\partial r}{\partial x}$  and  $\phi_\omega \frac{\partial r}{\partial \omega}$  are of second order, and can be neglected in Equation (3.26). This results in

$$(3.27) \quad -\frac{\partial r}{\partial x} + \phi_y = 0$$

## APPENDIX D

### Derivation of the Basic Flow Velocity

The basic flow velocity  $\Phi_{B_x}$  is the velocity obtained from one-dimensional flow theory, and is expressed as a function of the duct radius at the cross section under consideration.

The continuity equation is written

$$(3.28) \quad r^2 \pi \rho V = r^{*2} \pi \rho^* a^*$$

where  $V$  is the absolute value of the velocity. Since  $\rho V$  is a function of  $V$ , it can be developed with respect to  $(V - a^*)$ . One then obtains

$$(3.29) \quad \rho V = \rho^* a^* - \frac{\rho^*}{a^*} (\gamma + 1) \frac{(V - a^*)^2}{2}$$

If one introduces the relation

$$(3.30) \quad \Delta r = r - r_0$$

the continuity equation is rewritten as

$$\left[ \rho^* a^* - \frac{\rho^*}{a^*} (\gamma + 1) \frac{(V - a^*)^2}{2} \right] \pi (r_0 + \Delta r)^2 = \rho^* a^* \pi r_0^2$$

If  $r$  is small, the higher order terms can be neglected so that

$$(3.31) \quad (V - a^*) = \left[ \frac{4a^{*2}}{\gamma + 1} \frac{\Delta r}{r_0} \right]^{\frac{1}{2}}$$

Since  $\Phi_{B_x} = \frac{V}{a^*} - 1$ , the basic flow velocity is expressed by

$$(3.32) \quad \Phi_{B_x} = \left[ \frac{4}{\gamma + 1} \frac{\Delta r}{r_0} \right]^{\frac{1}{2}}$$

## APPENDIX E

### Analytical Representation of the Constants $I_{0,\mu}$ , $K_{0,\mu}$ , $I_{1,\mu}$ , and $K_{1,\mu}$

An analytical representation is sought for the integrals that occur in the expressions  $I_{0,\mu}$  and  $K_{0,\mu}$ . The first term  $I_{0,\mu}$  is

$$(3.33) \quad I_{0,\mu} = \int_0^1 \gamma Z_{0,\mu}^2(\gamma) d\gamma = \int_0^1 \gamma J_0^2(\bar{\eta}_{0,\mu} \gamma) d\gamma$$

Integrating<sup>3</sup>

$$(3.34) \quad I_{0,\mu} = \frac{\gamma^2}{2} \left[ J_0^2(\bar{\eta}_{0,\mu} \gamma) - J_0(\bar{\eta}_{0,\mu} \gamma) J_{-1}(\bar{\eta}_{0,\mu} \gamma) \right]_{\gamma=0}^{\gamma=1}$$

---

<sup>3</sup>Ibid., p. 145

Then<sup>4</sup> since

$$J_{-1}(\bar{\eta}_{0\mu} \gamma) = J_1(\bar{\eta}_{0\mu} \gamma)$$

$$(3.35) \quad I_{0\mu} = \frac{1}{2} [J_0^2(\bar{\eta}_{0\mu} \gamma) + J_1^2(\bar{\eta}_{0\mu} \gamma)]$$

The values for the arguments of the Bessel function  $J_0(\bar{\eta}_{0\mu} \gamma)$  were chosen so that  $\frac{dJ_0(\gamma)}{d\gamma} = 0$ . However, it is also true that  $\frac{dJ_0(\gamma)}{d\gamma} = J_1(\gamma)$ , so that Equation (3.35) simplifies to the analytical expression

$$(3.36) \quad I_{0\mu} = \frac{1}{2} J_0^2(\bar{\eta}_{0\mu} \gamma)$$

The term  $K_{0\mu}$  is defined by

$$(3.37) \quad K_{0\mu} = \int_0^1 \gamma^3 Z_{0\mu}(\gamma) d\gamma = \int_0^1 \gamma^3 J_0(\bar{\eta}_{0\mu} \gamma) d\gamma$$

If one denotes the argument by  $u = (\bar{\eta}_{0\mu} \gamma)$  Equation (3.37) is rewritten as

$$K_{0\mu} = \int_0^{\bar{\eta}_{0\mu}} \frac{u^3}{\bar{\eta}_{0\mu}^3} J_0(u) \frac{du}{\bar{\eta}_{0\mu}}$$

or simply

$$(3.38) \quad K_{0\mu} = \frac{1}{\bar{\eta}_{0\mu}^4} \int_0^{\bar{\eta}_{0\mu}} u^3 J_0(u) du$$

An integration<sup>5</sup> gives the relationship

$$(3.39) \quad K_{0\mu} = \frac{1}{\bar{\eta}_{0\mu}^4} [u^3 J_1(u) - 2 \int_0^{\bar{\eta}_{0\mu}} u^2 J_1(u) du]$$

or

$$(3.40) \quad K_{0\mu} = \frac{1}{\bar{\eta}_{0\mu}^4} [u^3 J_1(u) + 2 u^2 J_0(u) - 4 u J_1(u)]^{\bar{\eta}_{0\mu}}$$

If the Equation (3.40) is rewritten in terms of  $y$ , and the limits of  $y = 0$  and  $y = 1$  are inserted, the new relation becomes

$$(3.41) \quad K_{0\mu} = \frac{1}{\bar{\eta}_{0\mu}^4} [2 \bar{\eta}_{0\mu}^2 J_0(\bar{\eta}_{0\mu} \gamma) + (\bar{\eta}_{0\mu}^3 - 4 \bar{\eta}_{0\mu}) J_1(\bar{\eta}_{0\mu} \gamma)]$$

Since  $\frac{dJ_0(\gamma)}{d\gamma} = J_1(\gamma) = 0$  the  $K_{0\mu}$  is reduced to

$$(3.42) \quad K_{0\mu} = \frac{2}{\bar{\eta}_{0\mu}^2} J_0(\bar{\eta}_{0\mu} \gamma)$$

<sup>4</sup> Ibid., p. 145

<sup>5</sup> Ibid., p. 145

The term  $I_{\nu\mu}$  is defined as

$$(3.43) \quad I_{\nu\mu} = \int_0^1 \gamma \bar{z}_{\nu\mu}^2(\gamma) d\gamma = \int_0^1 \gamma J_\nu^2(\bar{\eta}_{\nu\mu} \gamma) d\gamma$$

It is seen that the integration<sup>6</sup> leads to the expression

$$(3.44) \quad I_{\nu\mu} = \frac{\nu^2}{2} \left[ J_\nu^2(\bar{\eta}_{\nu\mu} \gamma) - J_{\nu-1}(\bar{\eta}_{\nu\mu} \gamma) J_{\nu+1}(\bar{\eta}_{\nu\mu} \gamma) \right]'$$

Also<sup>7</sup>

$$(3.45) \quad \bar{z}_{\nu-1}(\gamma) = \frac{d\bar{z}_\nu(\gamma)}{d\gamma} + \frac{\nu}{\gamma} \bar{z}_\nu(\gamma)$$

$$(3.46) \quad -\bar{z}_{\nu+1}(\gamma) = \frac{d\bar{z}_\nu(\gamma)}{d\gamma} - \frac{\nu}{\gamma} \bar{z}_\nu(\gamma)$$

The substitution of Equations (3.45) and (3.46) into Equation (3.44) gives the new relation

$$(3.47) \quad I_{\nu\mu} = \frac{\nu^2}{2} \left[ J_\nu^2(\bar{\eta}_{\nu\mu} \gamma) + J_\nu'(\bar{\eta}_{\nu\mu} \gamma) \right]'$$

where  $J_\nu'(\bar{\eta}_{\nu\mu} \gamma)$  denotes the first derivative of  $J_\nu$  with respect to the argument.

Since the arguments are chosen such that  $J_\nu'(\eta_{\nu\mu} \gamma) = 0$  and the integration limits are  $\gamma = 0$  and  $\gamma = 1$ , the value of  $I_{\nu\mu}$  is

$$(3.48) \quad I_{\nu\mu} = J_\nu^2(\bar{\eta}_{\nu\mu} \gamma)$$

The term  $K_{\nu\mu}$  is written

$$(3.49) \quad K_{\nu\mu} = \int_0^1 \gamma^{\nu+1} \bar{z}_{\nu\mu}(\gamma) d\gamma = \int_0^1 \gamma^{\nu+1} J_\nu(\bar{\eta}_{\nu\mu} \gamma) d\gamma$$

Substituting the following differential equation

$$(3.50) \quad \frac{d}{d\gamma} \left[ \gamma J_\nu'(\bar{\eta}_{\nu\mu} \gamma) \right] + \left[ \gamma \bar{\eta}_{\nu\mu}^2 - \frac{\nu^2}{\gamma} \right] J_\nu(\bar{\eta}_{\nu\mu} \gamma) = 0$$

into Equation (3.49), one obtains

$$(3.51) \quad K_{\nu\mu} = \frac{1}{\bar{\eta}_{\nu\mu}^2} \left\{ - \int_0^1 \gamma^\nu \frac{d}{d\gamma} \left[ \gamma J_\nu'(\bar{\eta}_{\nu\mu} \gamma) \right] d\gamma + \nu^2 \int_0^1 \gamma^{\nu-1} J_\nu(\bar{\eta}_{\nu\mu} \gamma) d\gamma \right\}$$

<sup>6</sup> Ibid., p. 146

<sup>7</sup> Ibid., p. 145

Integrating by parts, one can rewrite Equation (3.51)

$$(3.52) \quad K_{\nu\mu} = \frac{1}{\bar{\eta}_{\nu\mu}^2} \left[ -\gamma^\nu \gamma J'_\nu(\bar{\eta}_{\nu\mu} \gamma) \right]_0^1 + \int_0^1 \gamma^\nu \gamma^\nu J'_\nu(\bar{\eta}_{\nu\mu} \gamma) d\gamma + \gamma^2 \int_0^1 \gamma^{\nu-1} J_\nu(\bar{\eta}_{\nu\mu} \gamma) d\gamma$$

Further integration and the substitution of the limits of  $\gamma = 0$  and  $\gamma = 1$ , leads to the relation

$$(3.53) \quad K_{\nu\mu} = \frac{1}{\bar{\eta}_{\nu\mu}^2} \left[ -J'_\nu(\bar{\eta}_{\nu\mu} \gamma) + \gamma J_\nu(\bar{\eta}_{\nu\mu} \gamma) \right]$$

Since  $J'_\nu(\bar{\eta}_{\nu\mu} \gamma) = 0$  the analytical representation of  $K_{\nu\mu}$  is

$$(3.54) \quad K_{\nu\mu} = \frac{\gamma}{\bar{\eta}_{\nu\mu}^2} J_\nu(\bar{\eta}_{\nu\mu} \gamma)$$

#### APPENDIX F

##### Solution of Equation (1.33) if $\Phi_{Bx}$ is Given by $x^*$

The solution of the differential equation (1.33) is determined for a more general case. Let the coefficients of the differential equation be such that it is a power  $t$  of the longitudinal position.

$$(3.55) \quad \Phi_{Bx} = x^*$$

This, when inserted into the differential equation gives

$$(3.56) \quad a_{0\mu}^{*''} + \frac{1}{x} a_{0\mu}^{*'} + \frac{1}{x^*} \left( \frac{\bar{\eta}_{0\mu}^2}{\gamma+1} \right) a_{0\mu}^* = 0$$

In order to remove the inconvenient power  $x^*$  in the coefficient of  $a_{0\mu}^*$ , let us introduce a new independent variable  $u$  by the equation  $u = x^k$  where  $k$  is a suitable constant which is chosen later. Then one obtains

$$(3.57) \quad \frac{da_{0\mu}^*}{dx} = k x^{k-1} \frac{da_{0\mu}^*}{du}$$

$$(3.58) \quad \frac{d^2 a_{0\mu}^*}{dx^2} = k^2 x^{2(k-1)} \frac{d^2 a_{0\mu}^*}{du^2} + k(k-1) x^{k-2} \frac{da_{0\mu}^*}{du}$$

If one writes  $\lambda = \frac{\bar{\eta}_{0\mu}^2}{\gamma+1}$  and inserts Equations (3.57) and (3.58)

into Equation (3.56), the following equation will result

$$(3.59) \quad \frac{d^2 a_{0\mu}^*}{du^2} + \gamma \frac{1}{u} \frac{da_{0\mu}^*}{du} + \frac{\lambda a_{0\mu}^*}{k^2 u^{(2+\frac{\gamma-2}{k})}}$$



where

$$(3.60) \quad \gamma = \frac{k+x-1}{k}$$

If  $k = 1 - \frac{x}{2}$ , Equation (3.59) simplifies to

$$(3.61) \quad \frac{d^2 a_{0,x}^*}{du^2} + \frac{\gamma}{u} \frac{da_{0,x}^*}{du} + \frac{\lambda}{k^2} a_{0,x}^* = 0$$

Equation (3.61) has the following form

$$(3.62) \quad a'' + \left( \frac{1-2s}{u} \right) a' + \left( \beta^2 + \frac{s^2-p^2}{u^2} \right) a = 0$$

the solution<sup>8</sup> of which is given by

$$(3.63) \quad a = u^s Z_p(\beta u)$$

where  $Z_p$  is a Bessel function of  $p$  order.

Let us set the following relations:

$$(3.64) \quad s = \frac{1-\gamma}{2}$$

$$(3.65) \quad p = s = \frac{1-\gamma}{2}$$

$$(3.66) \quad \beta = \left( \frac{\lambda}{k^2} \right)^{\frac{1}{2}}$$

The solution of the differential equation (3.61) involving coefficients that are powers of  $x$  is expressed as

$$(3.67) \quad a_{0,x}^* = u^s Z_s \left[ \left( \frac{\lambda}{k^2} \right)^{\frac{1}{2}} u \right]$$

where

$$(3.68) \quad s = \frac{1-x}{2-x}$$

---

<sup>8</sup> Ibid., p. 146

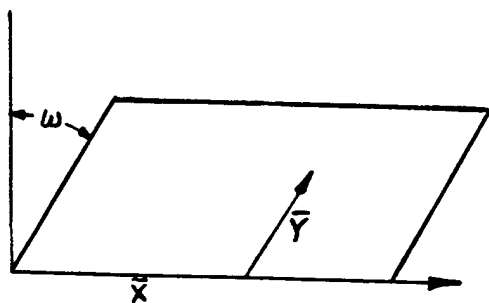


FIG. 1— CYLINDRICAL  
COORDINATE SYSTEM.

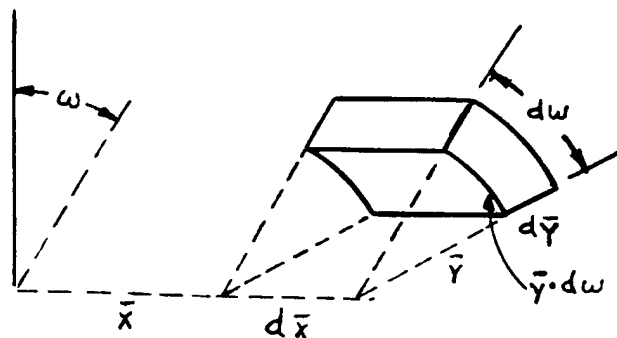


FIG. 2— SPACE ELEMENT IN  
CYLINDRICAL COORDINATE SYSTEM.

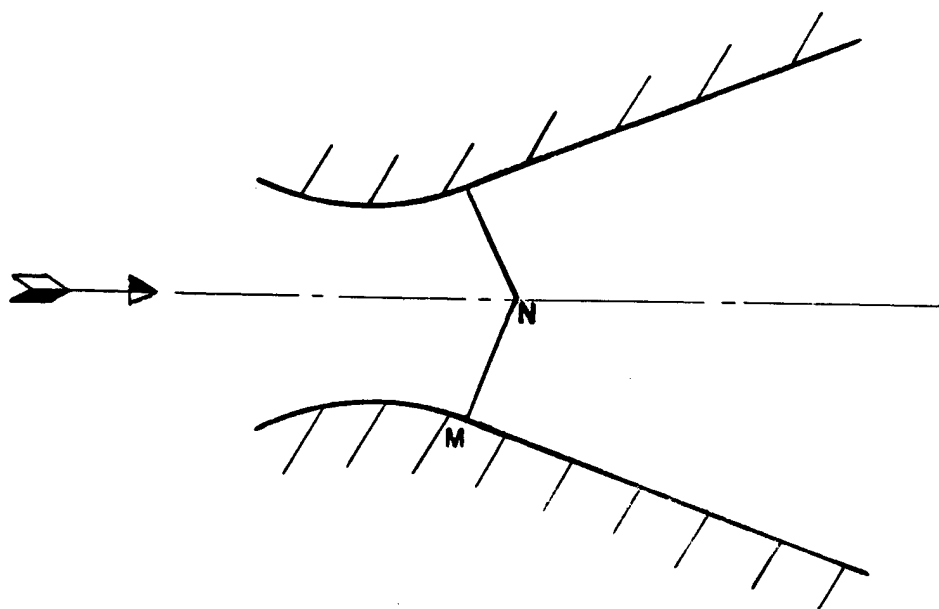


FIG. 3— PROPAGATION OF DISTURBANCES FROM SUPERSONIC JUNCTION.

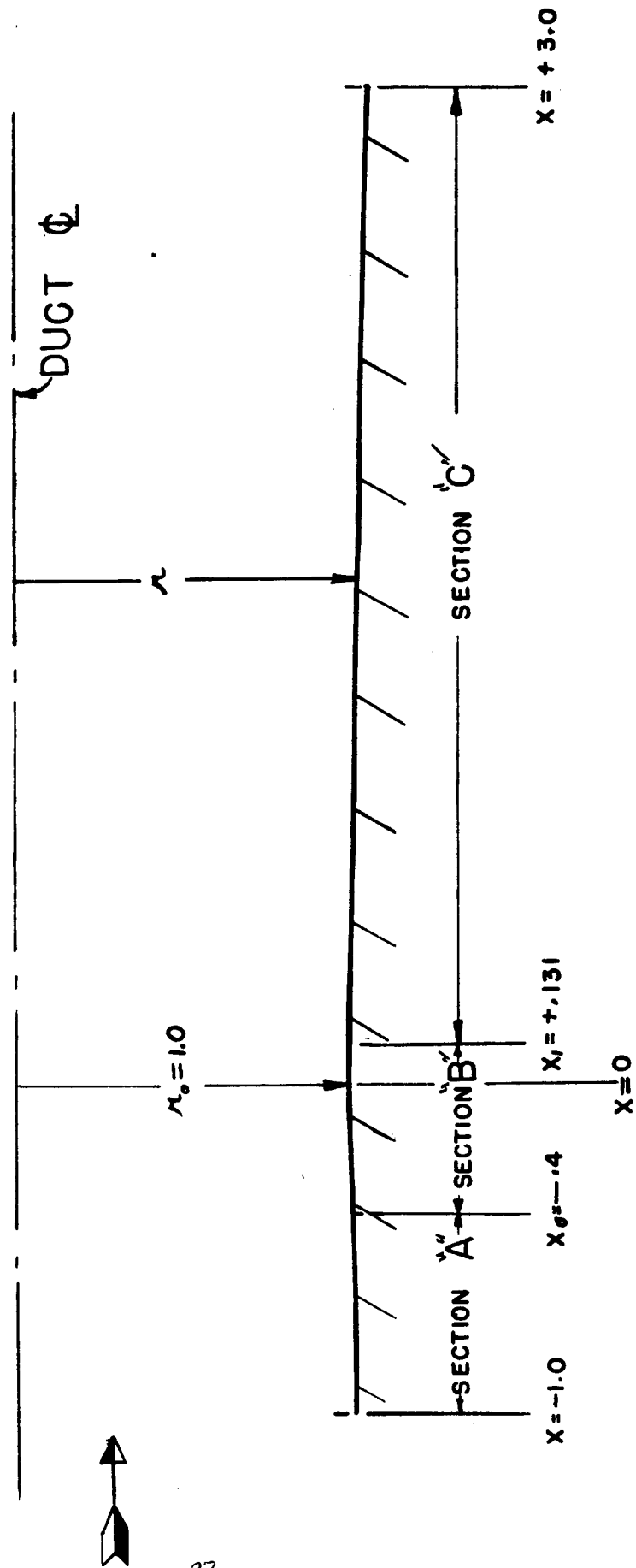
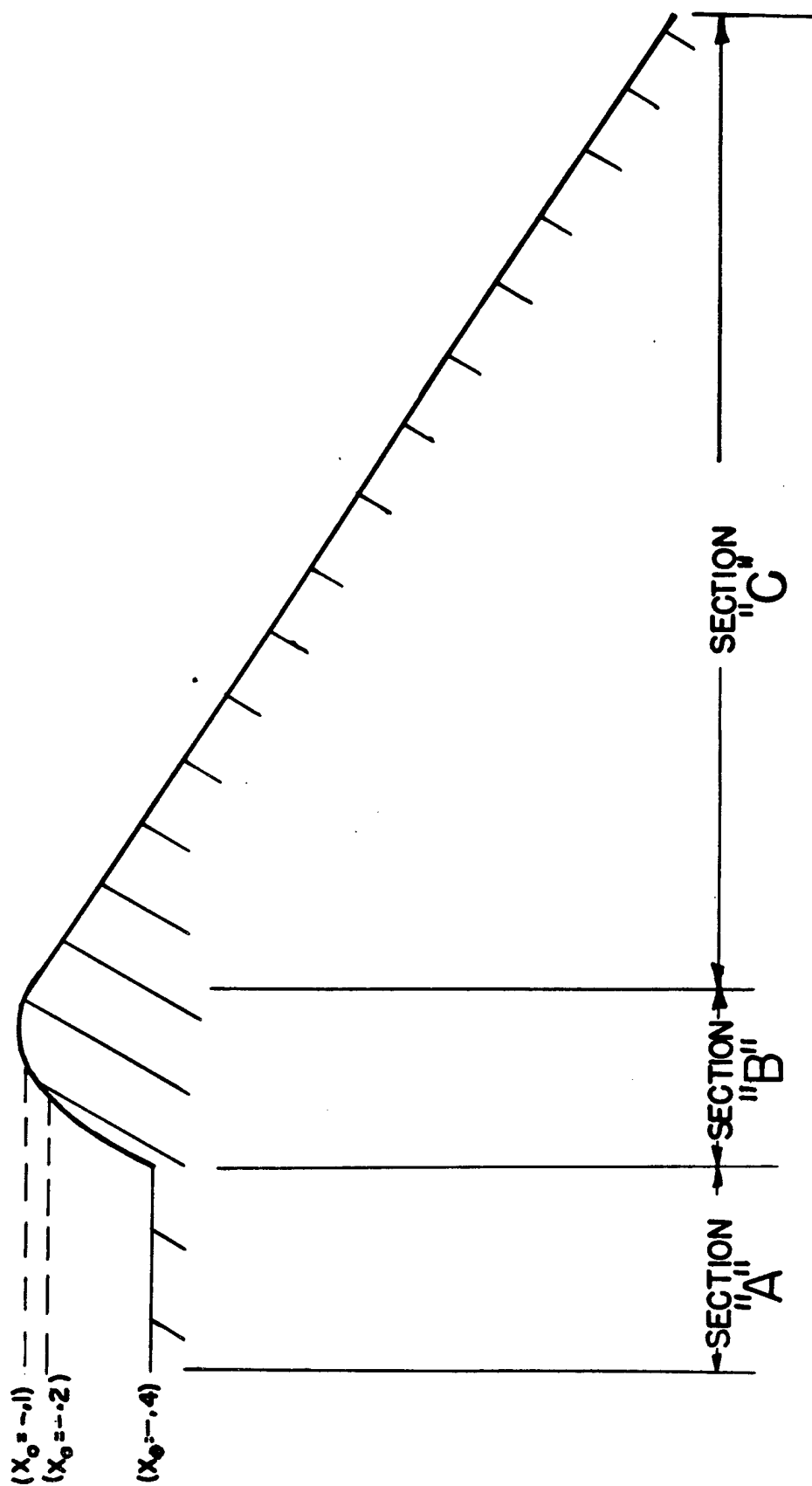


FIG.4— AXIALLY— SYMMETRIC DUCT CONTOUR

SCALE:— 1"= 2.5'



AF-TP-5790



SCALE:— 1"= 2.5' HORIZ.  
1"= 0.1' VERT.

FIG. 5— AXIALLY— SYMMETRIC-DUCT - CONTOUR

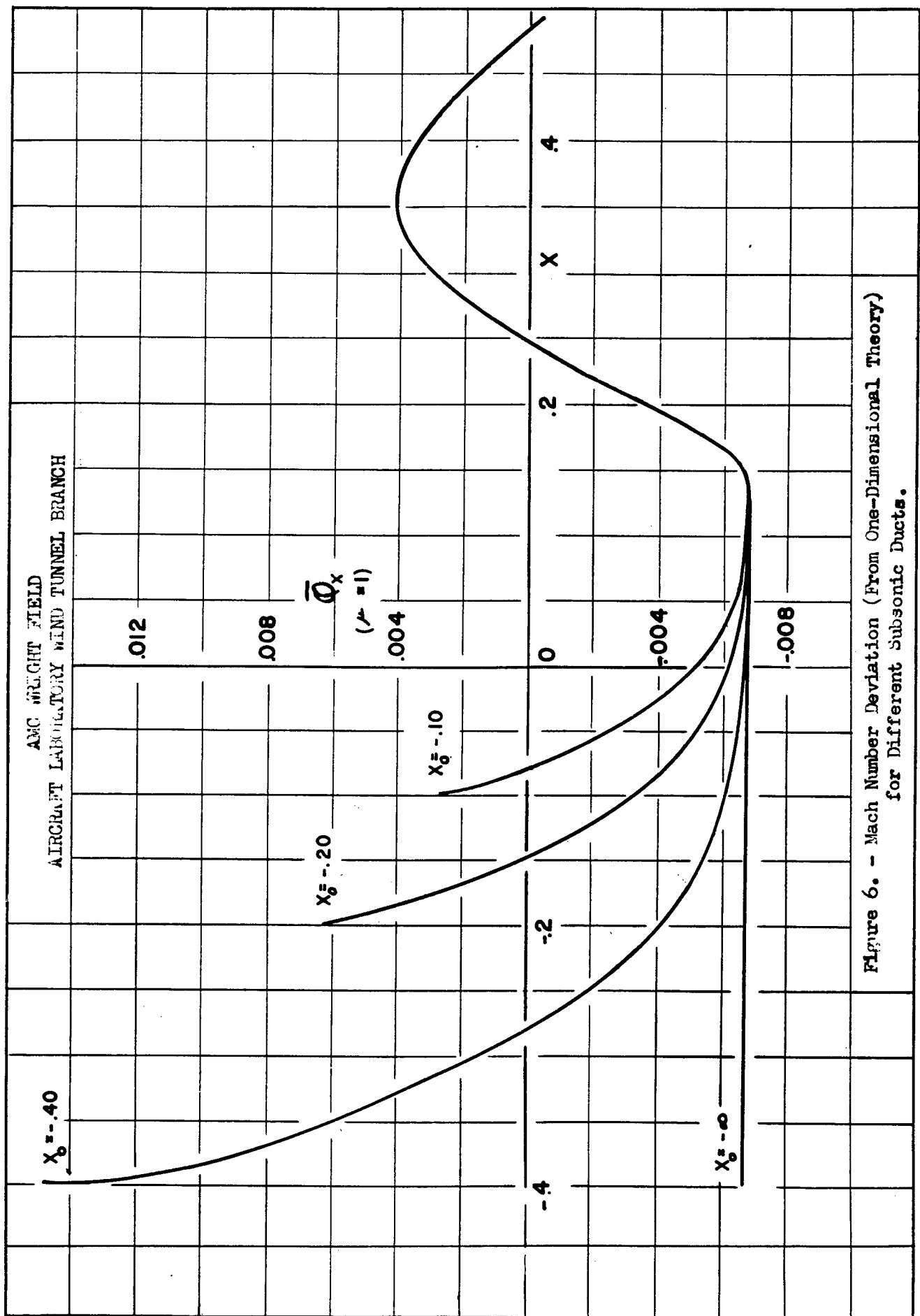


Figure 6. - Mach Number Deviation (From One-Dimensional Theory)  
for Different Subsonic Ducts.

AMC WRIGHT FIELD  
AIRCRAFT LABORATORY WIND TUNNEL BRANCH

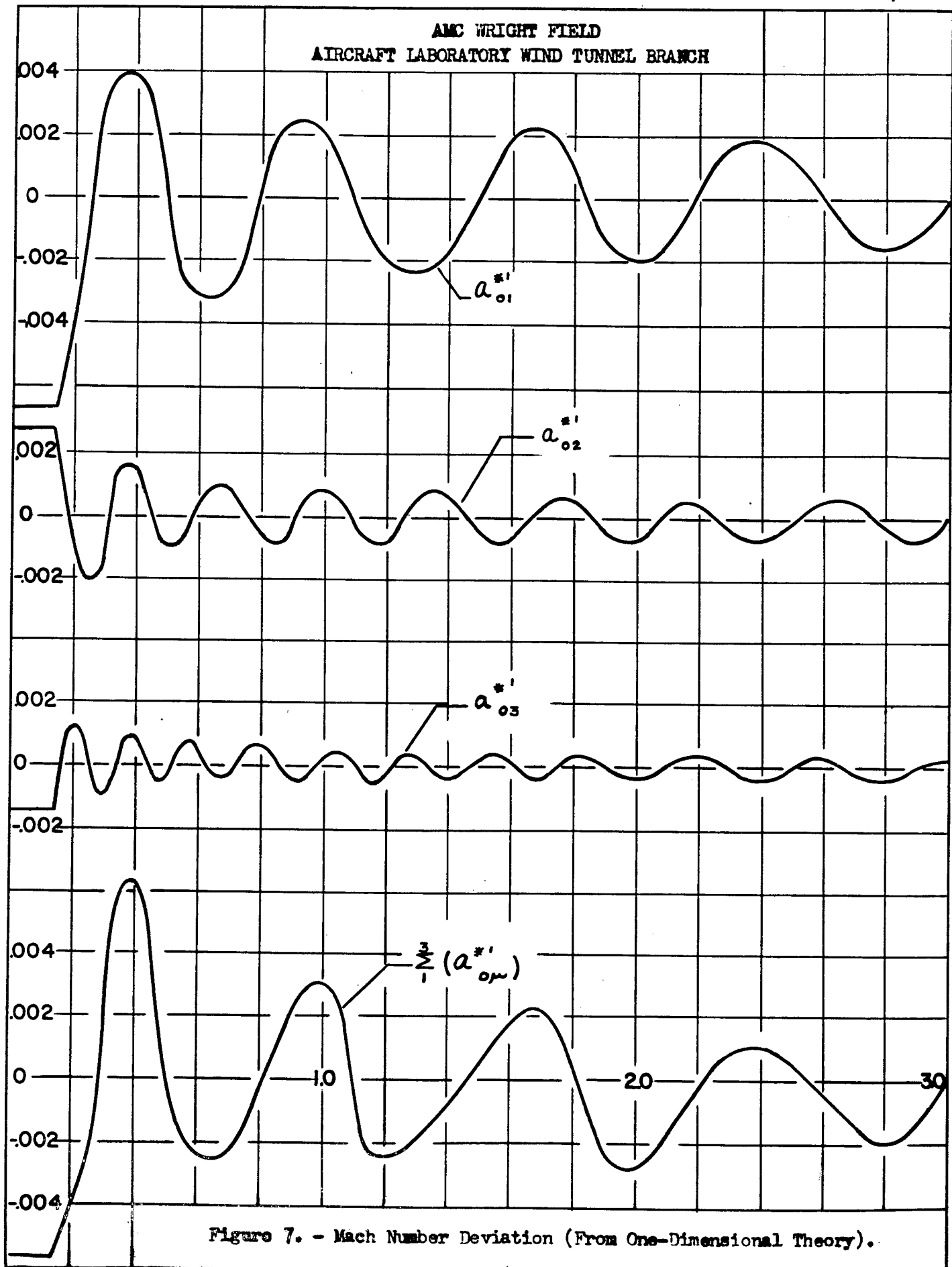


Figure 7. - Mach Number Deviation (From One-Dimensional Theory).

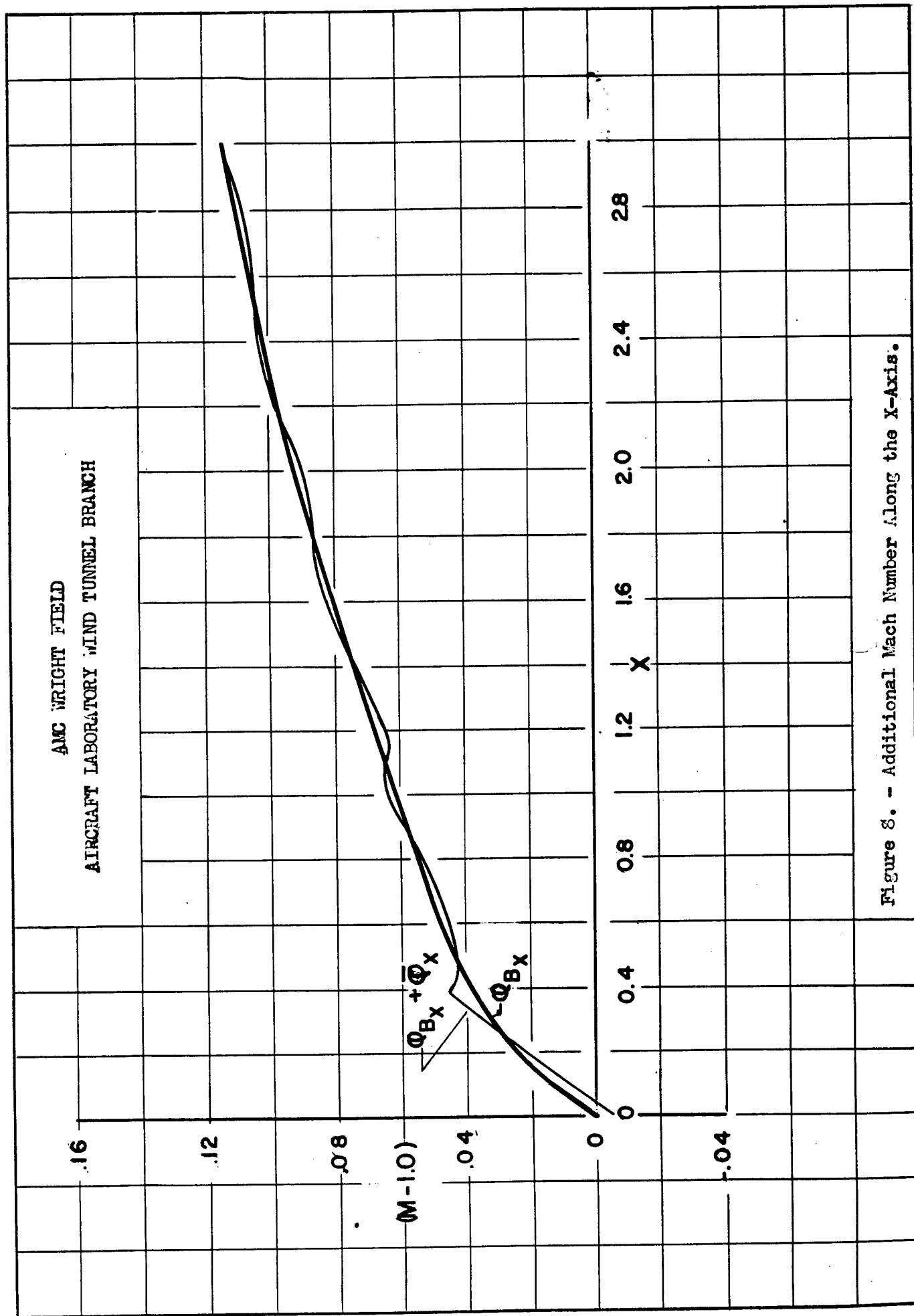


Figure 8. - Additional Mach Number Along the X-Axis.